Definitions about Relations

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November 3, 2017

5.1

(p. 197) A binary ***relation*** R from X to Y is a subset of $X \times Y$. X and Y are the ***domains*** of R.

5.3

(p. 202) The ***image*** of an element $a \in X$ under a relation R from X to Y is defined as

$$\mathcal{I}_R(a) = \{b \in Y \mid (a, b) \in R\}$$

(p. 202) The ***image*** of a set $A \subseteq X$ under a relation R from X to Y is defined as

$$\mathcal{I}_R(A) = \{b \in Y \mid \exists\ a \in A \mid (a, b) \in R\}$$

(p. 202) The ***inverse*** of a relation R from X to Y is the relation

$$R^{-1} = \{(b, a) \in Y \times X \mid (a, b) \in R\}$$

(p. 203) The ***composition*** of a relation R from X to Y and a relation S from Y to Z is the relation

$$S \circ R = \{(a, c) \in X \times Z \mid \exists\ b \in Y \mid (a, b) \in R \wedge (b, c) \in S\}$$

(p. 205, ex. 5.3.9) The ***identity relation*** on a set X is defined as

$$i_x = \{(x, x) \mid x \in X\}$$

5.4

(p. 205) A relation R on a set X is ***reflexive*** if every element is related to itself:

$$\forall\ x \in X, (x, x) \in R$$

(p. 206) A relation R on a set X is ***symmetric*** if for every pair in the relation, the inverse of the pair is also in the relation:

$$\forall\ x, y \in X,\ \text{if } (x, y) \in R \text{ then } (y, x) \in R$$

(p. 207) A relation R on a set X is ***transitive*** if any time one element is related to a second and the second is related to a third, then the first is also related to the third:

$$\forall\ x, y, z \in X,\ \text{if } (x, y) \in R \wedge (y, z) \in R \text{ then } (x, z) \in R$$

5.5

(p. 209) An ***equivalence relation*** is a relation that is reflexive, symmetric, and transitive.

(p. 211) For an equivalence relation R on X, let $[x]$ denote the image of a given $x \in X$ under R. We call $[x]$ the ***equivalence class*** of x under R.

(p. 211) Let X be a set, and let $P = \{X_1, X_2, \ldots X_n\}$ be a partition of X. Let R be the relation on X defined so that $(x, y) \in R$ if there exists $X_i \in P$ such that $x, y \in X_i$. We call R the ***relation induced*** by the partition.

5.7

(p.219) If R is a relation, then R^T is the ***transitive closure*** of R if

1. R^T is transitive;
2. $R \subseteq R^T$
3. If S is a relation such that $R \subseteq S$ and S is transitive, then $R^T \subseteq S$.

(p. 220) The ***reflexive closure*** and ***symmetric closure*** have similar definitions, replacing "transitive" with "reflexive" or "symmetric."

(p. 220) R^i is R composed with itself i times.

(p. 221) The ***iterated union*** of sets $A_1, A_2, \ldots, A_i$ is defined as

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$